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A Note on Definite Stochastic Sequential Machines[1]

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INTRODUCTION

A finite-state stochastic sequential machine, SSM, is a quadruple (X, Y, S, f) where X is a finite set of inputs, Y is a finite set of outputs, S is a finite set of states, and f is a conditional probability function from $S \times X \times S \times Y$ to $[0, 1]$ which is such that, for s and s' in S, x in X, y in Y,

$$f(s, x, s', y) \geqq 0$$

and

$$\sum_{s' \in S} \sum_{y \in Y} f(s, x, s', y) = 1$$

We interpret $f(s, x, s', y)$ as the conditional probability that the SSM will go to state s' and will produce the output y given that the SSM is in state s and receives the input x.

When we are only interested in the state-transition behavior of a SSM, we can represent the conditional probability function f by a set of stochastic matrices $\mathcal{P}$. Let the states in S be labeled $1, 2, \cdots, n$. Corresponding to each input x in X there is an $n \times n$ matrix in $\mathcal{P}$ such that the ijth entry of which is equal to $\sum_{y \in Y} f(i, x, j, y)$, that is, the conditional probability that the SSM will go to state j given that the SSM is in state i and receives the input x. We shall let $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{T}, \cdots$ denote the stochastic matrices in $\mathcal{P}$, which are also called the transitional probability matrices.

A SSM is said to be definite if there exists a positive integer k such that the product of any k matrices (not necessary distinct) in $\mathcal{P}$ is a matrix with identical rows. The least of such k's is called the order of the definite SSM. Clearly, for a definite SSM of order k, the state prob-

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ability distribution is independent of the initial state probability distribution after k or more transitions. In this paper, we study some of the properties of definite SSM's.

AUTONOMOUS STOCHASTIC SEQUENTIAL MACHINES

We limit our discussion in this section to SSM's with only one input in the set X. These machines are known as autonomous stochastic sequential machines. It follows that the state-transition behavior of an autonomous SSM is completely characterized by the (only) transitional probability matrix in $\mathcal{P}$. We shall denote this matrix $\mathbf{P}$.

Given the transitional probability matrix $\mathbf{P}$ of an autonomous SSM, we want to determine whether the SSM is definite. One straight-forward way of doing so is to compute $\mathbf{P}^2, \mathbf{P}^3, \mathbf{P}^4, \cdots$, and to see whether $\mathbf{P}^k$ is a matrix with identical rows. (One difficulty is that we might have to carry out the multiplication infinitely when the SSM is not definite. However, see Corollary 1.2 below.) Here, we shall present a test procedure which requires only a finite number of steps of computation, where the number of steps is bounded by the number of states of the SSM. We prove first a more general result (Theorem 1) on the rank of square matrices.

Let $\mathbf{P}$ be an $n \times n$ matrix. Let $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n$ denote the rows of $\mathbf{P}$, that is,

$$\mathbf{P} = \begin{bmatrix} \mathbf{r}_1 \\ \hdashline \mathbf{r}_2 \\ \hdashline \vdots \\ \hdashline \mathbf{r}_n \end{bmatrix}$$

Suppose that the rows of $\mathbf{P}$ are expressible as linear combinations of the rows $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_{n-t}$. We can write

$$\begin{aligned}
\mathbf{r}_1 &= \mathbf{r}_1 \\
\mathbf{r}_2 &= \mathbf{r}_2 \\
&\vdots \\
\mathbf{r}_{n-t} &= \mathbf{r}_{n-t} \\
\mathbf{r}_{n-t+1} &= g_1\mathbf{r}_1 + g_2\mathbf{r}_2 + \cdots + g_{n-t}\mathbf{r}_{n-t} \\
\mathbf{r}_{n-t+2} &= h_1\mathbf{r}_1 + h_2\mathbf{r}_2 + \cdots + h_{n-t}\mathbf{r}_{n-t} \\
&\vdots \\
\mathbf{r}_n &= m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + \cdots + m_{n-t}\mathbf{r}_{n-t}
\end{aligned}$$

Let $\mathbf{D}$ be the coefficient matrix of this set of equations, that is,

$$\mathbf{D} = \underbrace{\left[\begin{array}{cccc|c} & \mathbf{I} & & & \mathbf{0} \\ \hline g_1 & g_2 & \cdots & g_{n-t} & \\ h_1 & h_2 & \cdots & h_{n-t} & \\ & \vdots & & & \mathbf{0} \\ m_1 & m_1 & \cdots & m_{n-t} & \end{array}\right]}_{(n-t)\qquad\qquad t} \begin{array}{l} \} \, (n-t) \\ \\ \} \, t \end{array}$$

where $\mathbf{I}$ is the $(n - t) \times (n - t)$ identity matrix and the $\mathbf{0}$'s are zero matrices. We can then write $\mathbf{P} = \mathbf{DP}$. We define a *coefficient matrix*, $\mathbf{D}$, of $\mathbf{P}$ as the coefficient matrix of a set of equations that express the rows of $\mathbf{P}$ as linear combinations of a subset of the rows. Clearly, when the rows of $\mathbf{P}$ are expressed as linear combinations of $(n - t)$ of its rows, the corresponding coefficient matrix $\mathbf{D}$ will have t zero columns. If the rank of $\mathbf{P}$ is $(n - t)$, a coefficient matrix $\mathbf{D}$ will be called a *minimum coefficient matrix* if the rank of $\mathbf{D}$ is also $(n - t)$. It should be pointed out that, in general, a matrix does not have a unique coefficient matrix, nor does it have a unique minimum coefficient matrix. As an example, let

$$\mathbf{P} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 4 & 1 & 4 \\ 4 & 2 & -2 & 0 \\ 3 & 3 & 2 & 4 \end{bmatrix}$$

Because

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r}_1 \\ \mathbf{r}_2 &= \mathbf{r}_2 \\ \mathbf{r}_3 &= \mathbf{r}_3 \\ \mathbf{r}_4 &= \tfrac{3}{2}\mathbf{r}_1 - \tfrac{1}{2}\mathbf{r}_2 + \mathbf{r}_3 \end{aligned}$$

we have

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{3}{2} & -\frac{1}{2} & 1 & 0 \end{bmatrix}$$

as a coefficient matrix. Moreover, because the rank of $\mathbf{P}$ is 2 and

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r}_1 \\ \mathbf{r}_2 &= \mathbf{r}_2 \\ \mathbf{r}_3 &= -\mathbf{r}_1 + \mathbf{r}_2 \\ \mathbf{r}_4 &= \tfrac{1}{2}\mathbf{r}_1 + \tfrac{1}{2}\mathbf{r}_2 \end{aligned}$$

we have

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

as a minimum coefficient matrix.

Let $\mathbf{D}$ be a (minimum) coefficient matrix of an $n \times n$ matrix $\mathbf{P}$. If $\mathbf{D}$ contains t zero columns, $\mathbf{PD}$ will also contain t zero columns. If we delete the zero columns and their corresponding rows in the matrix $\mathbf{PD}$, we have an $(n - t) \times (n - t)$ submatrix, which will be called the *first order (minimum) reduced matrix* of $\mathbf{P}$ with respect to $\mathbf{D}$. Obviously, a reduced matrix of a nonsingular matrix is the matrix itself since only the identity matrix can be its coefficient matrix. In the previous example, for

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

we have

$$\mathbf{PD} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 4 & 1 & 4 \\ 4 & 2 & -2 & 0 \\ 3 & 3 & 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 7 & 0 & 0 \\ 6 & 7 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 3 & 7 & 0 & 0 \end{bmatrix}$$

and the minimum reduced matrix of $\mathbf{P}$ with respect to $\mathbf{D}$ is

$$\begin{bmatrix} 0 & 7 \\ 6 & 7 \end{bmatrix}$$

Let $\mathbf{P}_1$ denote a minimum reduced matrix of $\mathbf{P}$ (with respect to some minimum coefficient matrix $\mathbf{D}$). We can find a first order minimum reduced matrix of $\mathbf{P}_1$ (with respect to some minimum coefficient matrix $\mathbf{D}_1$) which will be denoted by $\mathbf{P}_2$ and called a *second order minimum reduced matrix* of $\mathbf{P}$. In a similar manner, we define a sequence of minimum reduced matrices of $\mathbf{P}$: $\mathbf{P}_3, \mathbf{P}_4, \cdots, \mathbf{P}_k$, called a *third, fourth,* $\cdots$, and kth *order minimum reduced matrix of* $\mathbf{P}$, respectively. We prove the following lemma and theorem:

LEMMA 1. *Let* $\mathbf{P}_1$ *be the first order reduced matrix of* $\mathbf{P}$ *with respect to a coefficient matrix* $\mathbf{D}$. *The rank of* $\mathbf{PD}$ *is the same as that of* $\mathbf{P}_1$.

Proof. Without loss of generality, we assume that the last t rows of **P** are expressible as linear combinations of the first $(n - t)$ rows. That is, the matrices **P** and **D** can be partitioned as

$$\mathbf{P} = \begin{array}{c} \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{CA} & \mathbf{CB} \end{array}\right] \begin{array}{l} \}(n-t) \\ \}t \end{array} \\ \begin{array}{cc} \underbrace{}_{(n-t)} & \underbrace{}_{t} \end{array} \end{array}$$

$$\mathbf{D} = \begin{array}{c} \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{C} & \mathbf{0} \end{array}\right] \begin{array}{l} \}(n-t) \\ \}t \end{array} \\ \begin{array}{cc} \underbrace{}_{(n-t)} & \underbrace{}_{t} \end{array} \end{array}$$

It follows that

$$\mathbf{PD} = \left[\begin{array}{c|c} \mathbf{A} + \mathbf{BC} & \mathbf{0} \\ \hline \mathbf{C}(\mathbf{A} + \mathbf{BC}) & \mathbf{0} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{P}_1 & \mathbf{0} \\ \hline \mathbf{CP}_1 & \mathbf{0} \end{array}\right]$$

Since the rows of $\mathbf{CP}_1$ are linear combinations of the rows of $\mathbf{P}_1$, the rank of **PD** is the same as that of $\mathbf{P}_1$. Q.E.D.

THEOREM 1. *For a square matrix* **P** *and any positive integer k, the rank of* $\mathbf{P}^{k+1}$ *is the same as that of* $\mathbf{P}_k$, *a kth order minimum reduced matrix of* **P**.

Proof. Without loss of generality, we assume that the last t rows of **P** are expressible as linear combinations of the first $(n - t)$ rows. Thus, **P** can be written as:

$$\mathbf{P} = \begin{array}{c} \left[\begin{array}{c} \mathbf{E} \\ \hline \mathbf{CE} \end{array}\right] \begin{array}{l} \}(n-t) \\ \}t \end{array} \\ \underbrace{}_{n} \end{array}$$

We have

$$\mathbf{P}^{k+1} = \mathbf{PP}^k = \mathbf{P}(\mathbf{DP})^k = (\mathbf{PD})^k\mathbf{P} = \left[\begin{array}{c|c} \mathbf{P}_1 & \mathbf{0} \\ \hline \mathbf{CP}_1 & \mathbf{0} \end{array}\right]^k \left[\begin{array}{c} \mathbf{E} \\ \hline \mathbf{CE} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \mathbf{P}_1^k & \mathbf{0} \\ \hline \mathbf{CP}_1^k & \mathbf{0} \end{array}\right] \left[\begin{array}{c} \mathbf{E} \\ \hline \mathbf{CE} \end{array}\right] = \left[\begin{array}{c} \mathbf{P}_1^k\mathbf{E} \\ \hline \mathbf{CP}_1^k\mathbf{E} \end{array}\right]$$

Therefore, the rank of $\mathbf{P}^{k+1}$ is the same as that of $\mathbf{P}_1{}^k\mathbf{E}$. However, since $\mathbf{E}$ is an $(n - t) \times n$ matrix of rank $(n - t)$, the rank of $\mathbf{P}_1{}^k\mathbf{E}$ is the same as that of $\mathbf{P}_1{}^k$.

To determine the rank of $\mathbf{P}_1{}^k$, we repeat the argument above. It follows that the rank $(\mathbf{P}^{k+1}) = \text{rank } (\mathbf{P}_1{}^k) = \text{rank } (\mathbf{P}_2^{k-1}) = \cdots = \text{rank } (\mathbf{P}_k)$. Q.E.D.

COROLLARY 1.1. *An autonomous* SSM *is definite of order k if corresponding to the transitional probability matrix* $\mathbf{P}$, $\mathbf{P}_k$ *is a* 1×1 *matrix and* $\mathbf{P}_{k-1}$ *is not.*

Proof. That $\mathbf{P}_k$ is a 1×1 matrix implies that the rank of $\mathbf{P}_{k-1}$ is 1. According to Theorem 1, the rank of $\mathbf{P}^k$ is also equal to 1. Since $\mathbf{P}^k$ is a stochastic matrix, no row in the matrix can possibly be a multiple of another row. Therefore, all the rows in $\mathbf{P}^k$ must be identical.

That $\mathbf{P}_{k-1}$ is not a 1×1 matrix implies that the rank of $\mathbf{P}_{k-2}$ is greater than 1. (If the rank of $\mathbf{P}_{k-2}$ is equal to 1, $\mathbf{P}_{k-1}$ will be a 1×1 matrix.) It follows that the rank of $\mathbf{P}^{k-1}$ is greater than 1 and not all the rows in $\mathbf{P}^{k-1}$ are identical. Q.E.D.

COROLLARY 1.2. *The order of an n-state definite autonomous* SSM *is less than or equal to* $(n - 1)$.[2]

Proof. Suppose that the order of the definite SSM is equal to k. The orders (sizes) of the matrices $\mathbf{P}, \mathbf{P}_1, \mathbf{P}_2, \cdots, \mathbf{P}_k$ must form a monotonically decreasing sequence. That is, $\mathbf{P}_1$ is at the most an $(n - 1) \times (n - 1)$ matrix, $\mathbf{P}_2$ is at the most an $(n - 2) \times (n - 2)$ matrix and so on. Because if the order of $\mathbf{P}_m$ $(m < k)$ is larger than 1 and is equal to the order of $\mathbf{P}_{m+1}$, the orders of $\mathbf{P}_{m+2}, \mathbf{P}_{m+3}, \cdots, \mathbf{P}_k$ will all be the same as that of $\mathbf{P}_m$, and the SSM is not definite. It follows that $k \leqq (n - 1)$. Q.E.D.

COROLLARY 1.3. For any positive integer m, the ranks of any two mth order minimum reduced matrices of a given matrix are the same.

[2] Blagoveshchensky (1960) has obtained the same result through an eigenvalue argument. Theorem 3 in the next section, due to Paz (1965), is a more general result.

Proof. Because the rank of any mth order minimum reduced matrix of $\mathbf{P}$ is equal to that of $\mathbf{P}^{m+1}$. Q.E.D.

COROLLARY 1.4. *If the rank of the transitional probability matrix of an n-state autonomous definite* SSM *is* $(n - 1)$, *the order of the* SSM *will also be* $(n - 1)$.

Proof. For any $n \times n$ matrix $\mathbf{P}$ of rank $(n - 1)$, the rank of $\mathbf{P}^2$ is larger than or equal to $(n - 2)$. If $\mathbf{P}$ is the transitional probability matrix of an autonomous definite SSM, the rank of $\mathbf{P}^2$ must be less than or equal to $(n - 2)$. Therefore, the rank of $\mathbf{P}^2$ must be $(n - 2)$. According to Theorem 1, the rank of $\mathbf{P}_1$, a minimum reduced matrix of $\mathbf{P}$, is also $(n - 2)$. Since $\mathbf{P}_1$ is an $(n - 1) \times (n - 1)$ stochastic matrix, the rank of $\mathbf{P}_1^2$, and thus the rank of $\mathbf{P}_2$ must be $(n - 3)$. Repeating this argument for $\mathbf{P}_2$ and so on, we prove the corollary. Q.E.D.

Corollaries 1.1 and 1.2 can be applied to determine whether an autonomous SSM is definite. According to Corollary 1.1, we can determine the rank of $\mathbf{P}^{k+1}$ by finding the reduced matrix $\mathbf{P}_k$. Moreover, in computing the sequence of reduced matrices $\mathbf{P}_1, \mathbf{P}_2, \cdots$ the occurrence of two successive reduced matrices $\mathbf{P}_m$ and $\mathbf{P}_{m+1}$ that are of the same order will mean that the SSM is not definite (unless the order of $\mathbf{P}_m$ is 1). As an example, consider an autonomous SSM with the transitional probability matrix.

$$\mathbf{P} = \begin{bmatrix} 0.3 & 0.1 & 0.5 & 0.1 \\ 0.3 & 0.1 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 & 0.1 \\ 0.2 & 0.0 & 0.5 & 0.3 \end{bmatrix}$$

We have (since $\mathbf{r}_4 = \mathbf{r}_1 + \mathbf{r}_2 - \mathbf{r}_3$ in $\mathbf{P}$)

$$\mathbf{P}_1 = \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.6 & 0.4 & 0.0 \\ 0.5 & 0.3 & 0.2 \end{bmatrix}$$

and (since $\mathbf{r}_3 = \frac{1}{2}\mathbf{r}_1 + \frac{1}{2}\mathbf{r}_2$ in $\mathbf{P}_1$)

$$\mathbf{P}_2 = \begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{bmatrix}$$

and (since $\mathbf{r}_2 = \mathbf{r}_1$ in $\mathbf{P}_2$)

$$\mathbf{P}_3 = [1]$$

Therefore, the SSM is definite and is of order 3.

If we elect to determine whether an autonomous SSM is definite by computing $\mathbf{P}^2, \mathbf{P}^3, \mathbf{P}^4, \cdots$, according to Corollary 1.2, we have to compute at the most up to $\mathbf{P}^{n-1}$ for an n state SSM. If the rows of $\mathbf{P}^{n-1}$ are not identical, we can conclude that the SSM is not definite.

EXTENSION TO THE GENERAL CASE

We shall extend the results in the previous section to the case in which the set $\mathcal{P}$ contains more than one stochastic matrix.

Let $\mathcal{P} = \{\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{T}, \cdots\}$. Let $\mathbf{D}$ be a coefficient matrix of *every* matrix in $\mathcal{P}$. The set $\{\mathbf{P}_1, \mathbf{Q}_1, \mathbf{R}_1, \mathbf{T}_1, \cdots\}$ which are the reduced matrices of the matrices in $\mathcal{P}$ with respect to $\mathbf{D}$ is called a set of *first order consistent reduced matrices of the set* $\mathcal{P}$. Similarly, for $i = 2, 3, \cdots$, let $\mathbf{D}_{i-1}$ be a coefficient matrix of every matrix in the set $\{\mathbf{P}_{i-1}, \mathbf{Q}_{i-1}, \mathbf{R}_{i-1}, \mathbf{T}_{i-1}, \cdots\}$. The set of matrices $\{\mathbf{P}_i, \mathbf{Q}_i, \mathbf{R}_i, \mathbf{T}_i, \cdots\}$ which are the reduced matrices of the matrices in $\{\mathbf{P}_{i-1}, \mathbf{Q}_{i-1}, \mathbf{R}_{i-1}, \mathbf{T}_{i-1}, \cdots\}$ with respect to $\mathbf{D}_{i-1}$ is called a set of ith *order consistent reduced matrives* of the set $\mathcal{P}$.

We prove the following theorem:

THEOREM 2. *The rank of the product of any sequence of k matrices (not necessarily distinct) from the set $\{\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{T}, \cdots\}$ is 1, if there exists a set of kth order consistent reduced matrices of $\{\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{T}, \cdots\}$ that are all 1×1 matrices.*

Proof. Consider the product of the two matrices $\mathbf{P}$ and $\mathbf{Q}$. Since $\mathbf{Q} = \mathbf{DQ}$, we have

$$\mathbf{PQ} = \mathbf{P}(\mathbf{DQ}) = (\mathbf{PD})\mathbf{Q}$$

According to Lemma 1, the rank of the matrix $\mathbf{PD}$ is the same as that of the matrix $\mathbf{P}_1$. Therefore, the rank of $\mathbf{PQ}$ is equal to or less than that of $\mathbf{P}_1$.

Similarly, consider the product of the matrices $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$. Since $\mathbf{Q} = \mathbf{DQ}$ and $\mathbf{R} = \mathbf{DR}$, we have

$$\mathbf{PQR} = \mathbf{P}(\mathbf{DQ})(\mathbf{DR}) = (\mathbf{PD})(\mathbf{QD})\mathbf{R}$$

Without loss of generality, let

$$\mathbf{D} = \begin{array}{c} \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{C} & \mathbf{0} \end{array}\right] \begin{array}{l} \}\,(n-t) \\ \}\,t \end{array} \\ \underbrace{\qquad}_{(n-t)}\underbrace{\quad}_{t} \end{array}$$

It follows that

$$\mathbf{PD} = \left[\begin{array}{c|c} \mathbf{P}_1 & \mathbf{0} \\ \hline \mathbf{CP}_1 & \mathbf{0} \end{array}\right]$$

$$\mathbf{QD} = \left[\begin{array}{c|c} \mathbf{Q}_1 & \mathbf{0} \\ \hline \mathbf{CQ}_1 & \mathbf{0} \end{array}\right]$$

and

$$(\mathbf{PD})(\mathbf{QD}) = \left[\begin{array}{c|c} \mathbf{P}_1\mathbf{Q}_1 & \mathbf{0} \\ \hline \mathbf{CP}_1\mathbf{Q}_1 & \mathbf{0} \end{array}\right]$$

Thus, the rank of $\mathbf{PDQD}$ is the same as that of $\mathbf{P}_1\mathbf{Q}_1$. Let $\mathbf{D}_1$ denote a coefficient matrix of the set of matrices $\{\mathbf{P}_1, \mathbf{Q}_1, \mathbf{R}_1, \mathbf{T}_1, \cdots\}$. Clearly, we have $\mathbf{D}_1\mathbf{Q}_1 = \mathbf{Q}_1$. Therefore, we have $\mathbf{P}_1\mathbf{Q}_1 = \mathbf{P}_1(\mathbf{D}_1\mathbf{Q}_1) = (\mathbf{P}_1\mathbf{D}_1)\mathbf{Q}_1$. However, according to Lemma 1, the rank of $\mathbf{P}_1\mathbf{D}_1$ is the same as that of $\mathbf{P}_2$. Therefore, the rank of $\mathbf{P}_1\mathbf{Q}_1$, and thus the rank of $\mathbf{PQR}$, is equal to or less than that of $\mathbf{P}_2$.

Similarly, consider the product of the matrices $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{T}$. Let

$$\mathbf{RD} = \left[\begin{array}{c|c} \mathbf{R}_1 & \mathbf{0} \\ \hline \mathbf{CR}_1 & \mathbf{0} \end{array}\right]$$

We have

$$\mathbf{PQRT} = \mathbf{R}(\mathbf{DQ})(\mathbf{DR})(\mathbf{DT}) = (\mathbf{PD})(\mathbf{QD})(\mathbf{RD})\mathbf{T}$$

and

$$(\mathbf{PD})(\mathbf{QD})(\mathbf{RD}) = \left[\begin{array}{c|c} \mathbf{P}_1\mathbf{Q}_1 & \mathbf{0} \\ \hline \mathbf{CP}_1\mathbf{Q}_1 & \mathbf{0} \end{array}\right](\mathbf{RD}) = \left[\begin{array}{c|c} \mathbf{P}_1\mathbf{Q}_1\mathbf{R}_1 & \mathbf{0} \\ \hline \mathbf{CP}_1\mathbf{Q}_1\mathbf{R}_1 & \mathbf{0} \end{array}\right]$$

Thus, the rank of $\mathbf{PQRT}$ is less than or equal to that of $\mathbf{P}_1\mathbf{Q}_1\mathbf{R}_1$. However, the rank of $\mathbf{P}_1\mathbf{Q}_1\mathbf{R}_1$ is less than or equal to that of $\mathbf{P}_2\mathbf{Q}_2$ which, in turn, is less than or equal to that of $\mathbf{P}_3$. Therefore, the rank of $\mathbf{PQRT}$ is less than or equal to that of $\mathbf{P}_3$.

For any sequence of k matrices $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{T}, \cdots$, we can repeat these arguments and show that the rank of the product $\mathbf{PQRT} \cdots$ is less than or equal to that of $\mathbf{P}_k$. Q.E.D.

According to this theorem, we have a *sufficient* test on whether a SSM is definite. If we are able to find a sequence of coefficient matrices $\mathbf{D}, \mathbf{D}_1, \mathbf{D}_2, \cdots, \mathbf{D}_{k-1}$ which are such that the corresponding kth order consistent reduced matrices of the transitional probability matrices are all 1×1 matrices, the SSM is definite. As an example, let

$$\mathbf{P} = \begin{bmatrix} .3 & .1 & .5 & .1 \\ .3 & .1 & .3 & .3 \\ .4 & .2 & .3 & .1 \\ .2 & .0 & .5 & .3 \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} .0 & .6 & .4 & .0 \\ .05 & .65 & .25 & .05 \\ .0 & .6 & .35 & .05 \\ .05 & .65 & .3 & .0 \end{bmatrix}$$

For

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

we have

$$\mathbf{P}_1 = \begin{bmatrix} .4 & .2 & .4 \\ .6 & .4 & .0 \\ .5 & .3 & .2 \end{bmatrix} \quad \text{and} \quad \mathbf{Q}_1 = \begin{bmatrix} .0 & .6 & .4 \\ .1 & .7 & .2 \\ .05 & .65 & .3 \end{bmatrix}$$

For

$$\mathbf{D}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

we have

$$\mathbf{P}_2 = \begin{bmatrix} .6 & .4 \\ .6 & .4 \end{bmatrix} \quad \text{and} \quad \mathbf{Q}_2 = \begin{bmatrix} .2 & .8 \\ .2 & .8 \end{bmatrix}$$

For

$$\mathbf{D}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

we have

$$\mathbf{P}_3 = [1] \quad \text{and} \quad \mathbf{Q}_3 = [1].$$

Thus, we conclude that **P** and **Q** are transitional probability matrices of a 4-state 2-input definite SSM.

THEOREM 3. *The order of a definite* SSM *with n states is equal to* $(n - 1)$ *or less.*

This theorem, due to Paz (1965), is a more general result than that stated in Corollary 1.2. Using this result, we shall prove Theorem 4. Note that, Theorem 2 gives only a sufficient test on whether a SSM is definite. The result in Theorem 4 gives a necessary and sufficient condition on a 3-state SSM being definite.

LEMMA 2. *Let* $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n$ *be the rows of an* $n \times n$ *stochastic matrix. If* $\mathbf{r}_n = g_1\mathbf{r}_1 + g_2\mathbf{r}_2 + \cdots + g_{n-1}\mathbf{r}_{n-1}$, *then* $\sum_{i=1}^{n-1} g_i = 1$.

Proof. Let **M** denote an $n \times 1$ column vector with 1's as all of its entries, that is,

$$\mathbf{M} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Because $\mathbf{r}_n\mathbf{M} = g_1\mathbf{r}_1\mathbf{M} + g_2\mathbf{r}_2\mathbf{M} + \cdots + g_{n-1}\mathbf{r}_{n-1}\mathbf{M}$, and because $\mathbf{r}_j\mathbf{M} = 1$ for $j = 1, \cdots, n$, we have $\sum_{i=1}^{n-1} g_i = 1$. Q.E.D.

THEOREM 4. *A 3-state* SSM *is definite if and only if there is a set of second order consistent reduced matrices of the transitional probability matrices which are* 1×1 *matrices.*

Proof. The sufficiency comes directly from Theorem 2. To prove the necessity, we suppose that $\{\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{T}, \cdots\}$ is a set of 3×3 stochastic matrices, in which the product of any two matrices (not necessarily distinct) contains three identical rows. We shall show that there exist coefficient matrices $\mathbf{D}$ and $\mathbf{D}_1$ which are such that every matrix in the set $\{\mathbf{P}_2, \mathbf{Q}_2, \mathbf{R}_2, \mathbf{T}_2, \cdots\}$ is a 1×1 matrix.

First of all, we see that for a matrix with three identical rows, its second order reduced matrix will be a 1×1 matrix for any choice of the coefficient matrices $\mathbf{D}$ and $\mathbf{D}_1$, as long as their ranks are 2 and 1, respectively. We shall consider, therefore, only those matrices in the set $\{\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{T}, \cdots\}$ that do not have three identical rows. Let

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$$

be two matrices in the set that do not have three identical rows. Since $\mathbf{P}^2$, $\mathbf{Q}^2$, $\mathbf{PQ}$, $\mathbf{QP}$ are all matrices with three identical rows, $\mathbf{P}$ and $\mathbf{Q}$ are both singular matrices. It is clear that, by renaming the states (and thus interchanging the rows and columns in the transitional probability matrices), we can express the third row as a linear combination of the first and the second rows in both $\mathbf{P}$ and $\mathbf{Q}$[3]. That is,

$$\begin{aligned} p_{31} &= k_1 p_{11} + k_2 p_{21} \qquad & q_{31} &= l_1 q_{11} + l_2 q_{21} \\ p_{32} &= k_1 p_{12} + k_2 p_{22} \qquad & q_{32} &= l_1 q_{12} + l_2 q_{22} \\ p_{33} &= k_1 p_{13} + k_2 p_{23} \qquad & q_{33} &= l_1 q_{13} + l_2 q_{23} \end{aligned}$$

where k_1, k_2, l_1, l_2, are constants. Since both $\mathbf{P}^2$ and $\mathbf{Q}^2$ are matrices with three identical rows, according to Theorem 1, the reduced matrix of $\mathbf{P}$ with respect to the coefficient matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k_1 & k_2 & 0 \end{bmatrix}$$

must have identical rows and the reduced matrix of $\mathbf{Q}$ with respect to the coefficient matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & 0 \end{bmatrix}$$

[3] If the first and second rows of $\mathbf{P}$ are identical and the first and third rows of $\mathbf{Q}$ are identical, no renaming of states is possible. However, it can be shown, in this case, that either $\mathbf{P}$ or $\mathbf{Q}$ must have three identical rows.

must also have identical rows. We thus have

$$\left.\begin{array}{ll} p_{11} + k_1 p_{13} = p_{21} + k_1 p_{23} & q_{11} + l_1 q_{13} = q_{21} + l_1 q_{23} \\ p_{12} + k_2 p_{13} = p_{22} + k_2 p_{23} & q_{12} + l_2 q_{13} = q_{22} + l_2 q_{23} \end{array}\right\} \tag{4.1}$$

We examine now the matrix **PQ**

$$\mathbf{PQ} = \begin{bmatrix} \sum_{i=1}^{3} p_{1i}q_{i1} & \sum_{i=1}^{3} p_{1i}q_{i2} & \sum_{i=1}^{3} p_{1i}q_{i3} \\ \sum_{i=1}^{3} p_{2i}q_{i1} & \sum_{i=1}^{3} p_{2i}q_{i2} & \sum_{i=1}^{3} p_{2i}q_{i3} \\ \sum_{i=1}^{3} p_{3i}q_{i1} & \sum_{i=1}^{3} p_{3i}q_{i2} & \sum_{i=1}^{3} p_{3i}q_{i3} \end{bmatrix}$$

Since, in the matrix **P**, the third row is equal to the sum of k_1 times the first row plus k_2 times the second row, it follows that, in the matrix **PQ**, the third row is also equal to the sum of k_1 times the first row and k_2 times the second row. According to Lemma 2, if the first two rows of **PQ** are identical, all the three rows of **PQ** are identical. Since, $\sum_{i=1}^{3} p_{1i}q_{i1} = \sum_{i=1}^{3} p_{2i}q_{i1}$ and $\sum_{i=1}^{3} p_{1i}q_{i2} = \sum_{i=1}^{3} p_{2i}q_{i2}$ imply that $\sum_{i=1}^{3} p_{1i}q_{i3} = \sum_{i=1}^{3} p_{2i}q_{i3}$, the conditions for **PQ** having three identical rows are

$$\left.\begin{array}{l} p_{11}q_{11} + p_{12}q_{21} + p_{13}(l_1 q_{11} + l_2 q_{21}) \\ \qquad = p_{21}q_{11} + p_{22}p_{21} + p_{23}(l_1 q_{11} + l_2 q_{21}) \\ p_{11}q_{12} + p_{12}q_{22} + p_{13}(l_1 q_{12} + l_2 q_{22}) \\ \qquad = p_{21}q_{12} + p_{22}q_{22} + p_{23}(l_1 q_{12} + l_2 q_{22}) \end{array}\right\} \tag{4.2}$$

Substituting the equations in (4.1) into the equations in (4.2), we have

$$\left.\begin{array}{l} (p_{13} - p_{23})[(l_1 - k_1)q_{11} + (l_2 - k_2)q_{21}] = 0 \\ (p_{13} - p_{23})[(l_1 - k_1)q_{12} + (l_2 - k_2)q_{22}] = 0 \end{array}\right\} \tag{4.3}$$

Since $k_1 + k_2 = l_1 + l_2 = 1$, that is $l_1 - k_1 = -(l_2 - k_2)$, the equations in (4.3) become

$$\left.\begin{array}{l} (l_1 - k_1)(p_{13} - p_{23})(q_{11} - q_{21}) = 0 \\ (l_1 - k_1)(p_{13} - p_{23})(q_{12} - q_{22}) = 0 \end{array}\right\} \tag{4.4}$$

The equations in (4.4) imply one (or more) of the following conditions:

(i) $l_1 = k_1$

(ii) $p_{13} = p_{23}$

(iii) $q_{11} = q_{21}$ and $q_{12} = q_{22}$

Suppose that (i) holds. Since $l_1 = k_1$ implies that $l_2 = k_2$, we can have

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k_1 & k_2 & 0 \end{bmatrix} \qquad \mathbf{D}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

as the coefficient matrices with respect to which the second order consistent reduced matrices $\mathbf{P}_2$, $\mathbf{Q}_2$ are 1×1 matrices.

Suppose that (ii) holds. According to the equations in (4.1), $p_{13} = p_{23}$ implies that $p_{11} = p_{21}$ and $p_{12} = p_{22}$. That is, the first and the second rows of $\mathbf{P}$ are identical. However, since the third row of $\mathbf{P}$ is a linear combination of the first and second row, according to Lemma 2 all three rows of $\mathbf{P}$ are identical. This is a contradiction to our assumption.

Suppose that (iii) holds. Since $q_{11} = q_{21}$ and $q_{12} = q_{22}$ imply that $q_{13} = q_{23}$, the three rows of $\mathbf{Q}$ will be identical, and this again is a contradiction to our assumption.

Repeating this argument for every pair of matrices in the set of matrices that do not have three identical rows we conclude that the second order reduced matrices of the set $\{\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{T}, \cdots\}$ with respect to the coefficient matrices:

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k_1 & k_2 & 0 \end{bmatrix} \qquad \mathbf{D}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

are all 1×1 matrices. Q.E.D.

CONCLUSION

The theory of definite automata was studied by Perles, Rabin, Shamir (1963), Liu (1963) and Paz (1965). In this paper, decision procedures for testing the definiteness of SSM's are proposed. The motivation of studying these procedures is not that much of a computational one. Rather, in this investigation we obtain results on the properties of definite SSM's that lead to further understanding of the state-transition behavior of this class of stochastic sequential machines. It is also interesting to point out the strong resemblance between the test procedures suggested here and the one suggested in Rabin (1963) and Liu (1963) for testing the definiteness of deterministic sequential machines in which rows of the state table of a sequential machine are combined to yield a sequence of "reduced state tables".

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